

Adjoint. Symmetric and Self-Adjoint Operators

We assume that \mathcal{H} is a Hilbert space. In this notes we will define self-adjoint unbounded operators and study its properties.

Definition 1. Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator densely defined (d.d.) i.e. $\overline{D(A)} = \mathcal{H}$.

We define the *adjoint* A^* of the operator A by

$$\begin{cases} D(A^*) = \{\eta \in \mathcal{H} : \exists \psi \in \mathcal{H} \text{ such that } (A\phi, \eta) = (\phi, \psi) \forall \phi \in D(A)\}, \\ A^*\eta = \psi. \end{cases}$$

Remarks 1.

1. A^* is well defined.

If there exists $\tilde{\psi} \in \mathcal{H}$ such that

$$(A\phi, \eta) = (\phi, \psi) = (\phi, \tilde{\psi}).$$

Then

$$(\phi, \psi - \tilde{\psi}) = 0 \text{ for all } \phi \in D(A)$$

which implies that $\psi \equiv \tilde{\psi}$ since $D(A)$ is dense in \mathcal{H} .

2. We have that

$$D(A^*) = \{\eta \in \mathcal{H} : L\eta : \phi \in D(A) \mapsto (A\phi, \eta) \text{ is continuous}\}. \quad (0.1)$$

Indeed, we can extend $\overline{L\eta} : \overline{D(A)} = \mathcal{H} \rightarrow \mathbb{C}$ continuous. i.e. $L\eta \in \mathcal{H}^*$ implies there exists $\psi \in \mathcal{H}$ such that

$$\overline{L\eta}(\phi) = (\phi, \psi) \quad \forall \phi \in \mathcal{H}$$

by the Riesz Theorem.

In particular, it holds that

$$(A\phi, \eta) = (\phi, \psi) \quad \forall \phi \in D(A).$$

3. It holds that

$$(A\phi, \eta) = (\phi, A^*\eta), \quad \forall \phi \in D(A), \quad \forall \eta \in D(A^*). \quad (0.2)$$

Verify that $A^* : D(A^*) \subset \mathcal{H} \rightarrow \mathcal{H}$ defines a linear operator.

Exercise 2. *If $A \in \mathcal{B}(\mathcal{H})$, show that $A^* \in \mathcal{B}(\mathcal{H})$ and it holds that*

$$(Af, g) = (f, A^*g) \quad \forall f, g \in \mathcal{H}$$

and

$$\|A\| = \|A^*\|.$$

Properties. Let $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : D(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be two linear operators d.d. Then it holds that

- (i) A^* is closed.
- (ii) $(\lambda A)^* = \bar{\lambda}A^*$ for all $\lambda \in \mathbb{C}$.
- (iii) $A \subseteq B$ implies that $B^* \subseteq A^*$.
- (iv) $A^* + B^* \subseteq (A + B)^*$.
- (v) $B^*A^* \subseteq (AB)^*$.
- (vi) $A \subseteq A^{**}$ where $A^{**} = (A^*)^*$.
- (vii) $(A + \lambda)^* = A^* + \bar{\lambda}$.

Theorem 3. *Let $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ a linear operator d.d., then it holds that*

- (i) A^* is closed;*
- (ii) A is closable if and only if A^* is d.d. in this case $\bar{A} = A^{**}$;*
- (iii) If A is closable, then $(\bar{A})^* = A^*$.*

Proof. To prove this theorem we will need some definitions and lemmas.

We start by considering the Hilbert space $\mathcal{H} \times \mathcal{H}$ equipped with the inner product

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle = (\phi_1, \phi_2)_{\mathcal{H}} + (\psi_1, \psi_2)_{\mathcal{H}}.$$

We define the operator

$$\begin{aligned} V : \mathcal{H} \times \mathcal{H} &\rightarrow \mathcal{H} \times \mathcal{H} \\ (\phi, \psi) &\mapsto V(\phi, \psi) = (-\psi, \phi). \end{aligned}$$

Notice that V is a unitary operator. In fact,

$$\begin{aligned} \langle V(\phi_1, \psi_1), V(\phi_2, \psi_2) \rangle &= \langle (-\psi_1, \phi_1), (-\psi_2, \phi_2) \rangle \\ &= (\psi_1, \psi_2) + (\phi_1, \phi_2) \\ &= \langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle. \end{aligned}$$

Lemma 1. *If $V : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary operator, then*

$$V(E^\perp) = V(E)^\perp, \quad \forall E \subseteq \mathcal{H},$$

where E^\perp denotes the orthogonal set to E which is defined as

$$E^\perp = \{\phi \in \mathcal{H} : (\phi, \eta) = 0, \forall \eta \in E\}.$$

Proof. Let $x \in E^\perp$ and $y = V(e) \in V(E)$, then

$$\langle V(x), y \rangle = \langle V(x), V(e) \rangle = \langle x, e \rangle = 0.$$

Thus $V(x) \in V(E)^\perp$ and so $V(E^\perp) \subset V(E)^\perp$.

Reciprocally, let $y \in V(E)^\perp$, since V is a bijection it implies there exists a unique $x \in \mathcal{H}$ such that $y = V(x)$.

If for all $e \in E$,

$$\langle x, e \rangle = \langle V(x), V(e) \rangle = 0$$

this implies that $x \in E^\perp$ and so $y \in V(E^\perp)$. Thus $V(E)^\perp \subset V(E^\perp)$.

This concludes the proof of the lemma. \square

Proof of (i) We denote by

$$G(A) = \{(x, Ax) : x \in D(A)\} \subseteq \mathcal{H} \times \mathcal{H}.$$

the graph of the operator A .

We can see that

$$\begin{aligned}(\phi, \eta) \in V(G(A))^\perp &\iff \langle (\phi, \eta), (-A\psi, \psi) \rangle = 0 \quad \forall \psi \in D(A) \\ &\iff -(\phi, A\psi) + (\eta, \psi) = 0 \quad \forall \psi \in D(A) \\ &\iff (\phi, A\psi) = (\eta, \psi) \quad \forall \psi \in D(A) \\ &\iff (A\psi, \phi) = (\psi, \eta) \quad \forall \psi \in D(A) \\ &\iff (\phi, \eta) \in G(A^*).\end{aligned}$$

This shows that $G(A^*) = \{V(G(A))\}^\perp$ is closed. (since the orthogonal of a set is always a closed subspace). Thus A^* is a closed operator.

Lemma 2. If $E \subseteq \mathcal{H}$, then $\overline{E} = (E^\perp)^\perp$.

Proof. If $e \in E$ and $x \in E^\perp$, then $(e, x) = 0$ and so $e \in (E^\perp)^\perp$. Hence $E \subset (E^\perp)^\perp$. Therefore $\overline{E} \subset (E^\perp)^\perp$.

Now we know that $\mathcal{H} = \overline{E} \oplus (\overline{E})^\perp$ because of the orthogonal projection Theorem.

On the other hand, $y \in (\overline{E})^\perp$ implies that $(y, x) = 0$ for all $x \in \overline{E}$ and this means that $y \in E^\perp$. Similarly, $y \in E^\perp$ implies that $(y, x) = 0$ for all $x \in E$ and so we have $(y, x) = 0$ for all $x \in (\overline{E})^\perp$. We deduce from this that $(\overline{E})^\perp = E^\perp$.

Thus

$$\mathcal{H} = \overline{E} \oplus E^\perp = E^\perp \oplus (\overline{E})^\perp.$$

We conclude that $\overline{E} = (E^\perp)^\perp$. □

Remark 1. *From the orthogonal projection Theorem we deduce that for any $F \subset \mathcal{H}$ closed and for all $x \in \mathcal{H}$, there exists a unique $y = \mathcal{P}_F(x) \in F$ such that $(x - \mathcal{P}_F(x), z) = 0$ for all $z \in F$, then $x - \mathcal{P}_F(x) \in F^\perp$, where $\mathcal{P}_F(x)$ denotes the projection on F at x .*

Since $G(A) \subset \mathcal{H} \times \mathcal{H}$, we have from Lemma 2, Lemma 1 and the fact that V is unitary that

$$\begin{aligned} \overline{G(A)} &= \{G(A)^\perp\}^\perp = \{V^2(G(A)^\perp)\}^\perp \\ &= \{V[V(G(A)^\perp)]\}^\perp = \{V[G(A^*)]\}^\perp. \end{aligned}$$

Thus,

$$\overline{G(A)} = \{V[G(A^*)]\}^\perp \tag{0.3}$$

and

$$G(A^*) = \{V[G(A)]\}^\perp \tag{0.4}$$

This means that A^* d.d. implies that A is closable.

Proof of (ii). If A^* is d.d then we deduce from (0.4) and (0.3) that

$$\{V[G(A^*)]\}^\perp = G(A^{**}) = \overline{G(A)}.$$

Then A is closable and $\overline{A} = A^{**}$.

Reciprocally, if $D(A^*)$ were not dense in \mathcal{H} , let $\psi \in D(A^*)^\perp$, $\psi \neq 0$, then $(\psi, 0) \in G(A^*)^\perp$ which implies that $(0, \psi) \in \{VG(A^*)\}^\perp = \overline{G(A)}$ but by Lemma 1.17 in [4] we have that $\overline{G(A)}$ is not the graph of any linear operator, that is, A is not a closable operator. This implies (ii).

Proof of (iii). Let A be a closable operator, since A^* is closed and (ii) holds we deduce that

$$A^* = \overline{A^*} = A^{***} = (A^{**})^* = (\overline{A})^*.$$

This completes the proof of Theorem 3. □

Example 1. We consider once again the operator A defined by

$$\begin{cases} D(A) = C^0([0, 1]) \subset L^2((0, 1)) \rightarrow L^2([0, 1]) \\ Af = \phi f(1) \text{ where } \phi \in L^2([0, 1]), \phi \neq 0. \end{cases}$$

In Example 1.21 in [4] we saw that the operator A is not closable. We shall show now that A^* is not densely defined.

Let $\eta \in \mathcal{H} = L^2([0, 1])$, then

$$(Af, \eta) = \int_0^1 f(1)\phi(x)\bar{\eta}(x) dx$$

Observe that the map

$$f \in C([0, 1]) \subset L^2([0, 1]) \mapsto f(1) \in \mathbb{C}$$

is not continuous in the L^2 topology. (To see this, take for instance $f_n(x) = x^n$. It is easy to check that $f_n \xrightarrow{L^2} 0$ and $f_n(1) = 1$.) As a consequence the only choice in order to have the map $f \mapsto (Af, \eta)$ continuous is taking $\eta \in L^2([0, 1])$ such that

$$\int_0^1 \phi(x) \bar{\eta}(x) dx = 0.$$

But this implies that

$$\begin{cases} D(A^*) = \{\phi\}^\perp \\ A^*\eta = 0. \end{cases}$$

In particular $\{\phi\}^\perp$ is not dense in $L^2([0, 1])$.

Application. Differentiable operators are closable

Consider a differentiable operator of order m ,

$$P(x, D) = \sum_{\substack{|\alpha| \leq m \\ \alpha \in \mathbb{N}^n}} a_\alpha(x) D_x^\alpha$$

where $a_\alpha \in C^\infty(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ is an open set.

We can define P_{\min} by

$$\begin{cases} D(P_{\min}) = C_0^\infty(\Omega) \subseteq L^2(\Omega) \\ P_{\min}\phi = P(x, D)\phi. \end{cases}$$

Taking $\phi, \psi \in C_0^\infty(\Omega)$, then

$$\begin{aligned}(P_{\min}\phi, \psi)_0 &= \left(\sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \phi, \psi \right)_0 \\ &= \int_\Omega \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \phi(x) \bar{\psi}(x) \, dx \\ &= \sum_{|\alpha| \leq m} \int_\Omega a_\alpha(x) \partial_x^\alpha \phi(x) \bar{\psi}(x) \, dx \\ &= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_\Omega \phi(x) \partial_x^\alpha (a_\alpha(x) \psi(x)) \, dx.\end{aligned}$$

This gives us

$$(P_{\min}\phi, \psi)_0 = (\phi, \zeta)_0$$

where $\zeta = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial_x^\alpha (\bar{a}_\alpha \psi)$.

Hence defining P_{\min}^* as

$$\begin{cases} D(P_{\min}^*) = C_0^\infty(\Omega), \\ P_{\min}^* \psi = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial_x^\alpha (\overline{a_\alpha} \psi). \end{cases}$$

This implies that P_{\min}^* is densely defined and from Theorem 3 it follows that P_{\min} is a closable operator.

Symmetric and Self-Adjoint Operators

Definition 2. Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator.

(i) We say that A is a **symmetric operator** if

$$(A\phi, \psi) = (\phi, A\psi), \quad \text{for all } \phi, \psi \in D(A).$$

This is equivalent to say that $A \subseteq A^*$.

(ii) We say that A is a **self-adjoint operator** if $A = A^*$.

Exercise 4. We say that a linear operator A is a **maximal symmetric operator** if

$$A \subseteq A^*$$

and

$$A \subseteq B, B \subseteq A^* \text{ then } A = B.$$

Prove that if $A = A^*$ then A is maximal symmetric.

Example 2. The operator H_0 is symmetric, i.e. $H_0 \subseteq H_0^*$. We recall that

$$\begin{cases} D(H_0) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \\ H_0 f = -\Delta f \end{cases}$$

For all $f, g \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} (H_0 f, g)_0 &= - \int_{\mathbb{R}^n} \Delta f(x) \overline{g(x)} dx \\ &= - \sum_{j=1}^n \int_{\mathbb{R}^n} \partial_{x_j}^2 f(x) \overline{g(x)} dx \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} f(x) \partial_{x_j}^2 \overline{g(x)} dx. \end{aligned}$$

Then

$$(H_0 f, g)_0 = (f, H_0 g)_0 \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

Now let $f, g \in H^2(\mathbb{R}^n)$, there exist $\{f_j\}, \{g_j\} \subseteq \mathcal{S}(\mathbb{R}^n)$ such that $f_j \xrightarrow{H^2} f$ and $g_j \xrightarrow{H^2} g$.

Notice that $\overline{C_0^\infty(\mathbb{R}^n)}^{H^2} = L^2(\mathbb{R}^n)$ (prove it!). Thus for any $j \in \mathbb{N}$ it holds that

$$(H_0 f_j, g_j)_0 = (f_j, H_0 g_j)_0$$

and making $j \rightarrow \infty$ it follows that

$$(H_0 f, g)_0 = (f, H_0 g)_0.$$

From the last identity we deduce that $H^2(\mathbb{R}^2) = D(H_0) \subseteq D(H_0^*)$ and $H_0 \subseteq H_0^*$.

Is H_0 a self-adjoint operator?

We need the following definition

Definition 3. Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$. Given a domain $\Omega \subset \mathbb{R}^n$, the **Sobolev space** $W^{k,p}(\Omega)$ is defined as,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\}.$$

We equipped the Sobolev space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}, & p = \infty. \end{cases}$$

It is usual to denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$ for it is a Hilbert space with the norm $W^{k,2}(\Omega)$.

Example 3. Consider the operators $A_j = \frac{1}{i} \frac{d}{dx}$, $j = 0, 1, 2$ with

$$D(A_0) = H^1([-\pi, \pi]) \subseteq L^2([-\pi, \pi]);$$

$$D(A_1) = \{\phi \in H^1([-\pi, \pi]) : \phi(-\pi) = \phi(\pi)\};$$

$$D(A_2) = \{\phi \in H^1([-\pi, \pi]) : \phi(-\pi) = \phi(\pi) = 0\}.$$

We will see that

- (i) A_1 and A_2 are symmetric operators.
- (ii) A_0 is not a symmetric operator.
- (iii) $A_2 = A_0^*$ ($\subseteq A_0$) this implies that $A_2^* = A_0^{**} = \overline{A_0} = A_0$ but $A_0 \not\subseteq A_2$ and so these operators are not self-adjoints.
- (iv) $A_1^* = A_1$.

Proof of (i). Let $\phi, \psi \in D(A_0)$ then

$$\begin{aligned}(A_0 \phi, \psi) &= \left(\frac{1}{i}\phi', \psi\right)_{L^2} = \frac{1}{i} \int_{-\pi}^{\pi} \phi'(x) \overline{\psi(x)} dx \\ &= \frac{1}{i} \phi(x) \overline{\psi(x)} \Big|_{-\pi}^{\pi} - \frac{1}{i} \int_{-\pi}^{\pi} \phi(x) \overline{\psi'(x)} dx \\ &= \frac{1}{i} [\phi(\pi) \overline{\psi(\pi)} - \phi(-\pi) \overline{\psi(-\pi)}] - \frac{1}{i} \int_{-\pi}^{\pi} \phi(x) \overline{\psi'(x)} dx\end{aligned}$$

This holds for absolutely continuous functions.

Then

$$(A_0 \phi, \psi) = \frac{1}{i} [\phi(\pi) \bar{\psi}(\pi) - \phi(-\pi) \bar{\psi}(-\pi)] + (\phi, A_0 \psi)_{L^2} \quad (0.5)$$

for all $\phi, \psi \in H^1([-\pi, \pi])$.

Therefore if $\phi, \psi \in D(A_j), j = 1, 2$, we deduce that

$$(A_j \phi, \psi)_{L^2} = (\phi, A_j \psi)_{L^2} \quad j = 1, 2,$$

which implies that $A_1 \subseteq A_1^*$ and $A_2 \subseteq A_2^*$.

Proof of (ii). In addition, there exist $\phi, \psi \in H^1([-\pi, \pi])$ such that

$$\phi(\pi) \overline{\psi(\pi)} \neq \phi(-\pi) \overline{\psi(-\pi)}$$

it yields

$$(A_0 \phi, \psi)_{L^2} \neq (\phi, A_0 \psi)$$

which implies that $A_0 \subsetneq A_0^*$.

Proof of (iii). Now let $\phi \in D(A_0)$ and $\psi \in D(A_2)$, then from (0.5) it follows that

$$(A_0\phi, \psi) = (\phi, A_2\psi)$$

which implies that $A_2 \subseteq A_0^*$.

To prove that $A_0^* \subseteq A_2$ it is enough to verify $D(A_0^*) \subseteq D(A_2)$.

Claim. If $\eta \in D(A_0^*)$, then

$$\int_{-\pi}^{\pi} A_0^* \eta(y) dy = 0. \quad (0.6)$$

Indeed, since $1 \in H^1([-\pi, \pi])$, it follows that

$$\int_{-\pi}^{\pi} \overline{A_0^* \eta(y)} dy = (1, A_0^* \eta) = (A_0(1), \eta) = 0.$$

Let $\eta \in D(A_0^*)$, we define $w(x) = i \int_{-\pi}^x A_0^* \eta(y) dy$. Then $w(-\pi) = w(\pi) = 0$ and so $w \in D(A_2)$ (prove it!).

Moreover,

$$A_2 w = \frac{1}{i} w'(x) = A_0^* \eta(x)$$

But since $A_2 \subseteq A_0^*$ we have $A_0^* w = A_2 w$. Thus

$$\begin{aligned} 0 &= (\phi, A_0^*(w - \eta)) \\ &= (A_0 \phi, w - \eta) \quad \forall \phi \in D(A_0). \end{aligned} \tag{0.7}$$

This implies that $w = \eta \in D(A_2)$ and so $D(A_0^*) \subseteq D(A_2)$.

To deduce the last affirmation we have used that

$$C_0^\infty([-\pi, \pi]) \subseteq \overline{(R(A_0))} = L^2([-\pi, \pi])$$

then taking $u \in C_0^\infty([-\pi, \pi])$ we have that $v(x) = \int_{-\pi}^x u(y) dy \in D(A_0)$ and $A_0 v = u$.

Proof of (iv). Notice that $A_2 = A_0^* \subseteq A_1 \subseteq A_0$. Then $A_0^* \subseteq A_1^* \subseteq A_0^{**} = \overline{A_0} = A_0$. Thus for all $\psi \in D(A_1^*)$,

$$A_1^* \psi = \frac{1}{i} \psi'.$$

From (ii) we already know that $A_1 \subseteq A_1^*$, we need to show now that $D(A_1^*) \subseteq D(A_1)$.

By the identity (0.5) it follows that for all $\phi \in D(A_1)$, and for all $\psi \in D(A_1^*)$,

$$(A_1 \phi, \psi)_{L^2} = (\phi, A_1^* \psi)$$

since $A_1^* \subseteq A_0$ and

$$\phi(\pi) \overline{\psi}(\pi) - \phi(-\pi) \overline{\psi}(-\pi) = \phi(\pi) (\overline{\psi}(\pi) - \overline{\psi}(-\pi)).$$

This implies that

$$\phi(\pi)(\overline{\psi}(\pi) - \overline{\psi}(-\pi)) = 0.$$

Choosing $\phi \equiv 1 \in D(A_1)$ it follows that $\psi(\pi) = \psi(-\pi)$, that is, $\psi \in D(A_1)$ that concludes the proof.

Remark 2. *From this example we can see that **it is not easy** at all to establish when a linear symmetric operator is self-adjoint just by using the definition. In what follow we will establish a criteria to determine when a symmetric operator is self-adjoint.*

Basic Criteria

The next result is an effective tool to determine when a symmetric operator is self-adjoint.

Theorem 5. *Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear operator d.d. such that $A \subseteq A^*$, then the following assertions are equivalent:*

- (i) $A = A^*$;
- (ii) A is closed and $\text{Ker}(A^* \pm i) = \{0\}$;
- (iii) $R(A \pm i) = \mathcal{H}$.

Remarks 6.

1. *The linear operator A does not have any special feature, i.e. the criteria holds for $\pm\lambda i$, $\lambda > 0$.*
2. *To prove (iii) \implies (ii) it is necessary to have $R(A + i) = \mathcal{H}$ and $R(A - i) = \mathcal{H}$.*

Proof.

(i) \implies (ii). If $A = A^*$, then A is closed.

Let now $\phi \in D(A^*) = D(A)$ such that $A^*\phi = i\phi$. Then

$$\begin{aligned} i\|\phi\|^2 &= (i\phi, \phi) = (A^*\phi, \phi) = (A\phi, \phi) \\ &= (\phi, A^*\phi) = (\phi, -i\phi) = -i\|\phi\|^2. \end{aligned}$$

This implies that $\phi = 0$, that is, $\text{Ker}(A^* - i) = \{0\}$.

Similarly, we will have $\text{Ker}(A^* + i) = \{0\}$.

□

(ii) \implies (iii). We will follow the following strategy:

1. We first prove that $R(A \pm i)^\perp = \{0\}$.
2. Then we show that $R(A \pm i)$ is closed.

Thus we can conclude that $R(A \pm i) = \mathcal{H}$. The latter follows from the orthogonal projection Theorem.

Affirmation. Let $B : D(B) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator d.d., then

$$R(B)^\perp = \text{Ker} B^*.$$

Proof of the Affirmation. (\subseteq) Let $\phi \in R(B)^\perp$, then for all $\psi \in D(B)$ we have that $(B\psi, \phi) = 0$. It follows that $\phi \in D(B^*)$ and $B^*\phi = 0$ which implies that $\phi \in \text{Ker} B^*$.

(\supseteq) Let $\phi \in \text{Ker} B^*$ then for all $\psi \in D(B)$

$$(B\psi, \phi) = (\psi, B^*\phi) = 0$$

we conclude that $\phi \in R(B)^\perp$. This completes the proof of the affirmation. □

Now we suppose (ii), we deduce from the affirmation that

$$R(A \pm i)^\perp = \text{Ker}(A \pm i)^* = \text{Ker}(A^* \mp i) = \{0\}$$

where we use property (vii) in page 2. The identity above gives us (1).

Next we shall show (2), i.e. $R(A \pm i)$ is closed.

Let $\{f_j\} \subset R(A \pm i)$ such that $f_j \xrightarrow{\mathcal{H}} f$. For all j there exist ϕ_j such that $f_j = (A \pm i)\phi_j$.

On the other hand, for all $\phi \in D(A)$ we have that

$$\begin{aligned}\|(A \pm i)\phi\|^2 &= ((A \pm i)\phi, (A \pm i)\phi) \\ &= (A\phi, A\phi) \pm i(\phi, A\phi) \mp i(\phi, A\phi) - i^2(\phi, \phi).\end{aligned}$$

Using that $A \subseteq A^*$ we conclude that

$$\|(A \pm i)\phi\|^2 = \|A\phi\|^2 + \|\phi\|^2. \tag{0.8}$$

From (0.8) we have that

$$\|f_j - f_l\|^2 = \|A(\phi_j - \phi_l)\|^2 + \|\phi_j - \phi_l\|^2 \quad \forall j, l. \quad (0.9)$$

Then we deduce from (0.9) that

$$\begin{cases} \|\phi_j - \phi_l\| \leq \|f_j - f_l\| \xrightarrow{j, l \rightarrow \infty} 0, \\ \|A(\phi_j - \phi_l)\| \leq \|f_j - f_l\| \xrightarrow{j, l \rightarrow \infty} 0. \end{cases}$$

Thus $\{\phi_j\}$ and $\{A\phi_j\}$ are Cauchy sequences in \mathcal{H} . Since \mathcal{H} is complete it follows that there exist $\phi, \psi \in \mathcal{H}$ such that

$$\begin{cases} \phi_j \xrightarrow{\mathcal{H}} \phi \\ A\phi_j \xrightarrow{\mathcal{H}} \psi \end{cases}$$

as $j \rightarrow \infty$.

Since A is closed it follows that $\phi \in D(A)$ and $\psi = A\phi$.

Thus

$$(A \pm i)\phi_j \xrightarrow{j \rightarrow \infty} (A \pm i)\phi \in R(A \pm i).$$

Therefore $R(A \pm i)$ is closed.

(iii) \implies (i). We already know that $A \subseteq A^*$. Left to show that $\overline{D(A^*)} \subseteq D(A)$.

Let $f \in D(A^*)$, by hypothesis $R(A - i) = \mathcal{H}$. Hence there exists $\phi \in D(A)$ such that

$$(A^* - i)f = (A - i)\phi \underset{A \subseteq A^*}{=} (A^* - i)\phi.$$

From this we conclude that

$$f - \phi \in \text{Ker}(A^* - i) = \text{Ker}(A + i)^* = R(A + i)^\perp = \{0\}$$

Thus $f = \phi \in D(A)$. Above we used the affirmation and the fact that $R(A + i) = \mathcal{H}$. □

Corollary 1 (Spectrum of a self-adjoint operator). *If $A = A^*$, then*

$$\sigma(A) \subseteq \mathbb{R}.$$

Proof. The basic criteria implies that for all $\lambda > 0$,

$$\begin{cases} \text{Ker}(A \pm i\lambda) = \{0\} \\ R(A \pm i\lambda) = \mathcal{H}. \end{cases}$$

From this we conclude that $\pm i\lambda \in \rho(A)$ for all $\lambda > 0$. If we denote $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, then $i\mathbb{R}^* \subseteq \rho(A)$.

Moreover, for all $\eta > 0$ we obtain

$$(A + \eta)^* = A^* + \bar{\eta} = A^* + \eta.$$

□

From this it follows then that

$$\begin{cases} \text{Ker}(A + \eta \pm i\lambda) = \{0\}, \\ R(A + \eta \pm i\lambda) = \mathcal{H}. \end{cases}$$

Hence $\eta \pm i\lambda \in \rho(A)$, for all $\eta \in \mathbb{R}$ and $\lambda > 0$. Therefore $\mathbb{C} \setminus \mathbb{R} \subset \rho(A)$.
In other words, $\sigma(A) \subset \mathbb{R}$. □

Definition 4. A linear symmetric operator ($A \subseteq A^*$) is called an **essentially self-adjoint** operator, if and only if, \overline{A} is self-adjoint. That is, $\overline{A}^* = A^* = \overline{A}$.

We can state another version of the Basic Criteria with the same proof as follows.

Theorem 7. Let A be a symmetric operator ($A \subseteq A^*$), then the following statements are equivalent.

- (i) $\overline{A} = A^*$ (\overline{A} is essentially self-adjoint);
- (ii) $\text{Ker}(A^* \pm i) = \{0\}$;
- (iii) $(A^* \pm i)$ is dense in \mathcal{H} .

Example 4. The operator $H_{0_{\min}}$ is essentially self-adjoint (prove it!).

Example 5. We will use the Basic Criteria to show that the operator

$$A_1 = \frac{1}{i} \frac{d}{dx}, \text{ with}$$

$$D(A_1) = \{\phi \in H^1([-\pi, \pi]) : \phi(-\pi) = \phi(\pi)\} \subseteq L^2([-\pi, \pi])$$

is a self-adjoint operator.

If $\phi, \psi \in D(A_1)$,

$$\begin{aligned} (A_1 \phi, \psi)_{L^2} &= \left(\frac{1}{i} \phi', \psi\right)_{L^2} = \frac{1}{i} \int_{-\pi}^{\pi} \phi'(x) \overline{\psi(x)} dx \\ &= \frac{1}{i} \phi(x) \overline{\psi(x)} \Big|_{-\pi}^{\pi} - \frac{1}{i} \int_{-\pi}^{\pi} \phi(x) \overline{\psi'(x)} dx \\ &= \frac{1}{i} [\phi(\pi) \overline{\psi(\pi)} - \phi(-\pi) \overline{\psi(-\pi)}] - \frac{1}{i} \int_{-\pi}^{\pi} \phi(x) \overline{\psi'(x)} dx \\ &= (\phi, A_1 \psi)_{L^2} \end{aligned}$$

This holds for absolutely continuous functions.

The last identity shows that A_1 is a linear symmetric operator, then we can apply the basic criteria to prove our claim. To do so we will verify that A_1 satisfies the condition (iii). That is,

$$R(A_1 \pm i) = L^2([-\pi, \pi]).$$

Indeed, given $f \in L^2([-\pi, \pi])$ we will show that there is $\phi \in D(A_1)$ such that

$$(A_1 \pm i)\phi = f.$$

In other words, we will try to solve the differential equations

$$\frac{1}{i} \frac{d}{dx} \phi \pm i\phi = f \tag{0.10}$$

for $\phi \in D(A_1)$.

As we observed in class we need to solve (0.10) for $-i$ and for i .

We consider first the case i in (0.10), thus

$$\frac{1}{i} \frac{d}{dx} \phi + i\phi = f$$

or

$$\phi' - \phi = if$$

whose solution is given by

$$\phi(x) = A e^x + i e^x \int_{-\pi}^x e^{-y} f(y) dy.$$

Next we choose A such that $\phi(-\pi) = \phi(\pi)$, that is,

$$A e^{-\pi} = A e^{\pi} + i e^{\pi} \int_{-\pi}^{\pi} e^{-y} f(y) dy$$

or

$$A = \frac{i e^{\pi}}{e^{-\pi} - e^{\pi}} \int_{-\pi}^{\pi} e^{-y} f(y) dy.$$

Hence,

$$\phi(x) = \left(\frac{ie^\pi}{e^{-\pi} - e^\pi} \int_{-\pi}^{\pi} e^{-y} f(y) dy \right) e^x + ie^x \int_{-\pi}^x e^{-y} f(y) dy. \quad (0.11)$$

Next we check that $\phi \in H^1([-\pi, \pi])$. We observe first that $e^x \in L^2([-\pi, \pi])$, then the first term on the right hand side of (0.11) is fine.

$$\begin{aligned} \int_{-\pi}^{\pi} \left| ie^x \int_{-\pi}^x e^{-y} f(y) dy \right|^2 dx &\leq \int_{-\pi}^{\pi} e^{2x} \left(\int_{-\pi}^x |e^{-y} f(y)| dy \right)^2 dx \\ &\leq \int_{-\pi}^{\pi} e^{2x} \left(\int_{-\pi}^x |e^{-y}|^2 dy \right) \left(\int_{-\pi}^x |f(y)|^2 dy \right) dx \\ &\leq c \|f\|_{L^2([-\pi, \pi])}^2. \end{aligned}$$

On the other hand, we have that

$$\phi'(x) = \left(\frac{ie^\pi}{e^{-\pi} - e^\pi} \int_{-\pi}^{\pi} e^{-y} f(y) dy \right) e^x + ie^x \int_{-\pi}^x e^{-y} f(y) dy + if(x).$$

It is not difficult to show that $\phi'(x) \in L^2([-\pi, \pi])$. This shows that

$$R(A_1 + i) = L^2([-\pi, \pi]).$$

A similar analysis proves that

$$R(A_1 - i) = L^2([-\pi, \pi]).$$

From the Basic Criteria we concludes that $A_1 = A_1^*$

References

- [1] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Volumes 1, 2
- [2] E. Hille, Methods in Classical and Functional Analysis
- [3] T. Kato, Perturbation Theory.
- [4] Notes on Unbounded Operators.